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A Geometrical Derivation of the Satellite Equations*

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I. INTRODUCTION

The advent of artificial earth satellites has renewed an interest in basic studies of the relevant equations of motion. Though the orbits of satellites have been studied by astronomers for many decades (indeed centuries), recent investigations suggest that few tools are available for treating associated engineering problems. The traditional procedures of astronomy are not sufficient for, perhaps, a number of reasons, but certainly one important source of difficulty stems from the demands of the astronautical engineers for coordinate systems entirely different from those adhered to by astronomers. Indeed, recent investigations present a picture of a somewhat frantic search for a coordinate system which is, on the one hand, suitable for engineering purposes and, on the other hand, adaptable to the highly developed perturbational procedures of astronomy. In the investigations to date, these dual objectives certainly have not been met though a number of important aspects of satellite motion have been revealed.

The typical approach [1, 2, 3, 4] has been to introduce an artificial "orbital plane" with a mean motion more or less matching that of the true orbital plane (a plane containing the velocity vector). It turns out that the artificial motion introduces extraneous and completely unnecessary complications while effectively removing any hope of a rigorous treatment of the equations. Other novel approaches, [5, 6, 7], suggest the use of known, exact non-Keplerian orbits. Though one might, conceivably, "match" a true orbit in this way, the coincidence would be but an accident (or fortunate circumstance) and the sum total of knowledge would be little changed. Again, a rigorous treatment of the equations appears to be out of the question in such a setting.

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In this brief paper we derive the equations of motion for a satellite relative to the true orbital plane. At each instant this plane contains the origin of the coordinate system, the satellite and the satellite velocity vector. Of course, this is the orbital plane of astronomy, but we abandoned completely the traditional concept of an osculating Keplerian orbit [8]. Rather, we choose a set of independent coordinates, suitable for engineering applications, which include the all important radial distance r. Once the coordinates are chosen, one could presumably derive the equations of motion using the classical method of Lagrange (variation of parameters) and there would be little to report. This is not done, however, for we wish to clarify the basic geometrical picture. Thus, the equations of motion in traditional spherical coordinates are transformed to the orbital plane coordinates and as a by-product of the transformation process, the true motion of the orbital plane, as a rigid body, is revealed. It seems strange that the geometrical picture of the motion of the orbital plane as a rigid body has never been really clarified.

We treat a completely general satellite motion which is induced by an arbitrary potential field, U. We envision a gravitational potential, but this restriction is only superficial and a matter of notational convenience for purposes of discussion. Since one is dealing with the orbital plane of astronomy, all the methods and results (perturbational or otherwise) of classical astronomy apply here to the extent that they would in the traditional astronomical system of variables. However, here one takes advantage of any symmetries of the potential and is thus able to reduce the order of the system in important special cases.

II. The Motion of the Orbital Plane of a Satellite

The equations of motion of a satellite in spherical coordinates (see Fig. 1, for notation) are [9],

$$\frac{d}{dt}\left(r^2 \sin^2 \theta \frac{d\Phi}{dt}\right) = \frac{\partial U}{\partial \Phi} \tag{1}$$

$$\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) - r^2 \sin\theta \cos\theta \left(\frac{d\Phi}{dt}\right)^2 = \frac{\partial U}{\partial \theta} \tag{2}$$

$$\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r \sin^2\theta \left(\frac{d\Phi}{dt}\right)^2 = \frac{\partial U}{\partial r} \tag{3}$$

where U is the gravitational potential.

For purpose of discussion, the plane $\theta = \pi/2$ is called the *equatorial* plane and the half line $\theta = 0$ is called the *polar axis*. It is convenient to introduce the auxiliary dependent variable

$$p = r^2 \sin^2 \theta \frac{d\Phi}{dt} \tag{4}$$

which is proportional to the angular momentum about the polar axis. Equation (1) then becomes, simply,

$$\frac{dp}{dt} = \frac{\partial U}{\partial \Phi} \tag{5}$$

If the potential U is symmetric with respect to the polar axis, then p is constant and (4) is a useful integral of the system. Quite generally, $p \neq 0$ [1] and so also $d\Phi/dt \neq 0$. Thus, Φ is a suitable independent variable and using (4) and (5), (2) may be written

$$\frac{d^2 \cot \theta}{d\Phi^2} + \cot \theta = -\frac{r^2 \sin^2 \theta}{p^2} \left[\frac{\partial U}{\partial \theta} + \frac{d \cot \theta}{d\Phi} \frac{\partial U}{\partial \Phi} \right] \tag{6}$$

The equations

$$\cot \theta = \tan \alpha \sin (\Phi + \Omega) \tag{7}$$

$$\sin \beta = \csc \alpha \cos \theta \tag{8}$$

$$\cos \beta = \sin \theta \cos (\Phi + \Omega) \tag{9}$$

express the pertinent geometrical relationships between a plane which contains the radius vector along r (i.e., any plane which contains the origin and the satellite) and the equatorial plane. For the orbital plane, (7), (8), and (9) become identities and the radius vector must rotate in the orbital plane. The latter is an evident geometrical implication of the fact that the satellite velocity vector must always lie in the orbital plane. Thus, the variations (if any) of α and Ω must depict a *rigid rotation* of the orbital plane about the radius vector along r. This rotation couples the α and Ω variations according to the equations

$$\frac{d\Omega}{dt} = \omega \frac{\sin \beta}{\sin \alpha} \tag{10}$$

$$\frac{d\alpha}{dt} = -\omega \cos \beta \tag{11}$$

[1] Polar orbits require $p = 0$ and must be obtained as limits of the nonpolar orbits discussed here.

which are readily obtained from the geometry. Here ω denotes the instantaneous rate of rotation of the orbital plane about the radius vector along r, as indicated in Fig. 1. Using (4), (10) and (11) may be expressed in the alternate forms

$$\frac{d\Omega}{d\Phi}=\frac{\omega}{p}\,\frac{r^2\sin^2\theta\sin\beta}{\sin\alpha} \tag{12}$$

$$\frac{d\alpha}{d\Phi}=-\frac{\omega}{p}\,r^2\sin^2\theta\cos\beta \tag{13}$$

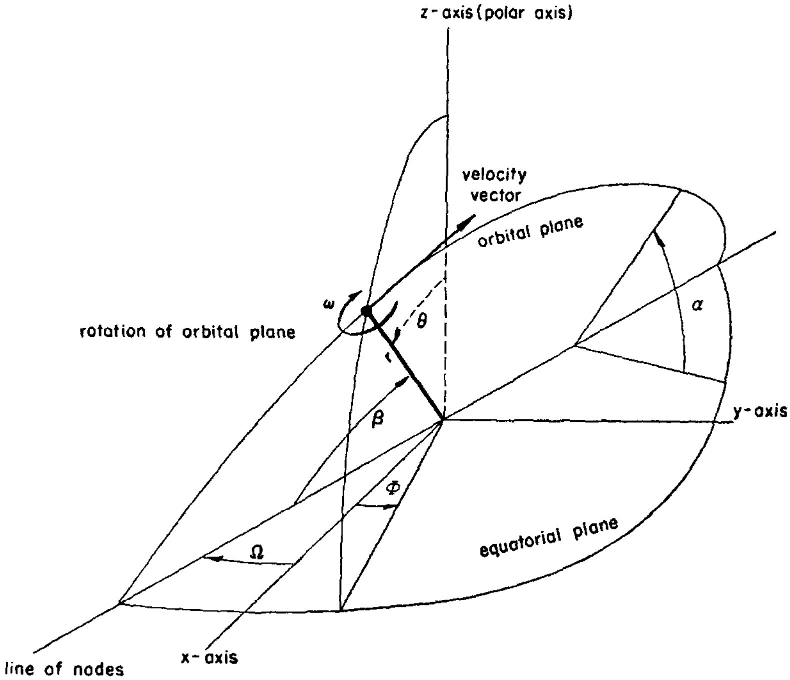

FIG. 1

We now reduce (6) to a simple algebraic equation for ω. Using (7), (8), (9), (12), and (13), it is found that

$$\frac{d\cot\theta}{d\Phi}=\tan\alpha\cos(\Phi+\Omega)\left(1+\frac{d\Omega}{d\Phi}\right)+\sec^2\alpha\sin(\Phi+\Omega)\frac{d\alpha}{d\Phi} \tag{14}$$

$$=\tan\alpha\cos(\Phi+\Omega)$$

which is obtained for either a stationary or a rotating orbital plane. Differentiating (14), we have,

$$\frac{d^2 \cot \theta}{d\Phi^2} = -\cot\theta\left(1 + \frac{d\Omega}{d\Phi}\right) + \sec^2\alpha \cos(\Phi + \Omega)\frac{d\alpha}{d\Phi}$$

which, in turn, yields

$$\frac{d^2 \cot \theta}{d\Phi^2} + \cot\theta = -\frac{\omega}{p}\frac{r^2 \sin^3\theta}{\cos^2\alpha} \tag{15}$$

Thus, according to (14) and (15), (6) becomes

$$\omega = \frac{\cos^2\alpha}{p \sin\theta}\left[\frac{\partial U}{\partial\theta} + \tan\alpha\frac{\cos\beta}{\sin\theta}\frac{\partial U}{\partial\Phi}\right] \tag{16}$$

Using (10), (11), and (16), we have immediately the following expressions:

$$\frac{d\Omega}{dt} = \frac{\cos^2\alpha}{p \sin\alpha}\frac{\sin\beta}{\sin\theta}\left[\frac{\partial U}{\partial\theta} + \tan\alpha\frac{\cos\beta}{\sin\theta}\frac{\partial U}{\partial\Phi}\right] \tag{17}$$

$$\frac{d\alpha}{dt} = -\frac{\cos^2\alpha}{p}\frac{\cos\beta}{\sin\theta}\left[\frac{\partial U}{\partial\theta} + \tan\alpha\frac{\cos\beta}{\sin\theta}\frac{\partial U}{\partial\Phi}\right] \tag{18}$$

III. The Radial Motion of a Satellite

Owing to the simplified form of Eq. (14), the radial equation (3) assumes a form identical to that for a stationary orbital plane. Indeed, it is easily verfied that

$$r\left(\frac{d\theta}{dt}\right)^2 + r\sin^2\theta\left(\frac{d\Phi}{dt}\right)^2 = \frac{p^2}{r^3\cos^2\alpha}$$

which is obtained for either a stationary or a rotating orbital plane. Thus (3) becomes

$$\frac{d^2 r}{dt^2} - \frac{p^2}{r^3\cos^2\alpha} = \frac{\partial U}{\partial r} \tag{19}$$

The four first-order equations (4), (5), (17), and (18) together with the single second-order equation (19) constitute a system equivalent to the three original second-order equations (1), (2), and (3). In typical cases, the orbital plane is either stationary or nearly so and perturbational procedures are generally feasible. To this end, one attempts to introduce β, the central angle in the orbital plane as independent variable. This is usually appropriate with β steadily increasing, though in one instance [2] it has led to a fictitious geometry and suprious analysis. If Eq. (8) is differentiated to yield

$$\cos\beta\,\frac{d\beta}{dt} = \csc\alpha\sin^3\theta\,\frac{d\cot\theta}{d\Phi}\,\frac{d\Phi}{dt} - \csc^2\alpha\cos\alpha\cos\theta\,\frac{d\alpha}{dt}$$

and (4), (8), (9), (14), and (18) are used, one obtains

$$\frac{d\beta}{dt} = \frac{p\sec\alpha}{r^2} + \frac{\cos^3\alpha\cos\theta}{p\sin^2\alpha\sin\theta}\left[\frac{\partial U}{\partial\theta} + \tan\alpha\,\frac{\cos\beta}{\sin\theta}\,\frac{\partial U}{\partial\Phi}\right] \tag{20}$$

Thus, if $d\beta/dt \neq 0$, (5), (17), and (18) may be expressed in the forms:

$$\frac{dp}{d\beta} = \frac{r^2\,\dfrac{\partial U}{\partial\Phi}}{p\sec\alpha + \dfrac{\cos^3\alpha\cos\theta\,r^2}{p\sin^2\alpha\sin\theta}\left[\dfrac{\partial U}{\partial\theta} + \tan\alpha\,\dfrac{\cos\beta}{\sin\theta}\,\dfrac{\partial U}{\partial\Phi}\right]} \tag{21}$$

$$\frac{d\Omega}{d\beta} = \frac{r^2\cos^3\alpha\cos\theta\left[\dfrac{\partial U}{\partial\theta} + \tan\alpha\,\dfrac{\cos\beta}{\sin\theta}\,\dfrac{\partial U}{\partial\Phi}\right]}{p^2\sin^2\alpha\sin\theta + r^2\cos^4\alpha\cos\theta\left[\dfrac{\partial U}{\partial\theta} + \tan\alpha\,\dfrac{\cos\beta}{\sin\theta}\,\dfrac{\partial U}{\partial\Phi}\right]} \tag{22}$$

$$\frac{d\alpha}{d\beta} = -\frac{r^2\cos^3\alpha\sin^2\alpha\cos\beta\left[\dfrac{\partial U}{\partial\theta} + \tan\alpha\,\dfrac{\cos\beta}{\sin\theta}\,\dfrac{\partial U}{\partial\Phi}\right]}{p^2\sin^2\alpha\sin\theta + r^2\cos^4\alpha\cos\theta\left[\dfrac{\partial U}{\partial\theta} + \tan\alpha\,\dfrac{\cos\beta}{\sin\theta}\,\dfrac{\partial U}{\partial\Phi}\right]} \tag{23}$$

and with

$$u = \frac{1}{r} \tag{24}$$

(19) becomes

$$k\sec\alpha\,\frac{d}{d\beta}\left(k\sec\alpha\,\frac{du}{d\beta}\right) + \sec^2\alpha\,u = -\frac{r^2}{p^2}\,\frac{\partial U}{\partial r} \tag{25}$$

where

$$k = 1 + \frac{\cos^4\alpha}{p^2\sin^2\alpha}\,r^2\,\frac{\cos\theta}{\sin\theta}\left[\frac{\partial U}{\partial\theta} + \tan\alpha\,\frac{\cos\beta}{\sin\theta}\,\frac{\partial U}{\partial\Phi}\right] \tag{26}$$

For a central force field,[2] p, α, Ω, and $k = 1$ are each constant so that (25) alone remains. The latter is integrable in closed form for arbitrary $U = U(r) = U(1/u)$. For the case of a Newtonian potential $U = G/r$, integration of (25) yields the familiar Keplerian orbit

$$\frac{1}{r} = u = \frac{G \cos^2 \alpha}{p^2} [1 + e \cos (\beta - \beta_0)]$$

where e and β_0 are arbitrary constants. More generally (21), (22), (23), and (25) constitute[3] a fifth-order system depicting the motion of the orbital plane as a rigid body and the motion of the satellite with respect to the orbital plane. The motion (if any) of the orbital plane at any instant is a rotation about the radius vector along r at the rate ω given in (16), while the radius vector itself rotates in the orbital plane. It should be noted that changes in the central angle β are affected by *both* rotations and so $d\beta/dt$ is not in general the rate of rotation of the radius vector. Rather the rate of rotation of the radius vector is given by the first term on the right in (20).

IV. Satellite Motion for Symmetric Potentials

In many important cases the potential U is independent of the longitude Φ (or is assumed to be) and, as indicated previously, one integral, $p =$ const. is immediately available. In addition, Eqs. (23) and (25) become independent of Ω (the position of the line of nodes) and hence of the differential equation (22). Thus, rather fortuitously the system reduces to the third order, a result apparently not generally known. Of course, to locate the orbital plane, subsequent to the resolution of (23) and (25), one must perform the quadrature called for in (22). It is rather interesting to find that the radial motion of the satellite is independent of the precessional motion of the orbit about the polar axis, inasmuch as the recent theories have concentrated so very much upon the latter, novel feature of earth satellites. However, it is now clear that the radial motion of a satellite may be determined without knowledge of any precessional motion of the orbital plane. The precessional motion of the orbital plane arises only incidentally and bears no dynamical significance.

Finally, we note that if $\partial U/\partial\theta$ is proportional to $1/r^2$ and otherwise independent of r, the system further reduces. For then Eq. (23) becomes independent of u and thus determines α uniquely in terms of β, the

[2] The potential U is a function of r only, i.e., U possesses spherical symmetry.

[3] Note that θ is given in terms of α and β by (8).

independent variable. With $\alpha = \alpha(\beta)$ so determined, (25) becomes a second-order equation in u alone. Since k, given by (26), also becomes explicit in β, nonlinearities in (25) will arise only from the potential on the right. For example, if the potential is of the form[4]

$$U = \frac{G}{r} + \frac{G_1(\theta)}{r^2}$$

then, for arbitrary $G_1(\theta)$, (25) becomes a *linear* second order equation with (in general) nonconstant coefficients.

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[4] Recent satellite theories based on "intermediary orbits" [5, 6], concern potentials of this form.